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# THE FUNCTIONAL EQUATION FOR POINCARÉ SERIES OF TRACE RINGS OF GENERIC $2\times 2$ MATRICES

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ABSTRACT

In this note we give a rational expression for the Poincaré series of $\Pi_{m,2}$, the trace ring of $m$ generic $2\times 2$ matrices. This result extends the computations of E. Formanek for $m\leq 4$. As a consequence, we prove that the Poincaré series satisfies the functional equation

$$\mathcal{P}(\Pi_{m,2};1/t)=-t^{4m}\cdot\mathcal{P}(\Pi_{m,2},t)\qquad(m>2)$$

supporting the conjecture that $\Pi_{m,2}$ is a Gorenstein ring.

## 1. The rational expression

Throughout this note, $k$ will be a field of characteristic zero. By $R$ we will denote the polynomial ring $k[x_{ij}(l);\ 1\leq i,j\leq n;\ 1\leq l\leq m]$. The sub-$k$-algebra of $M_n(R)$ generated by the elements $\{X_l=(x_{ij}(l))_{i,j}\}$ is $\mathbf{G}_{m,n}$, the ring of $m$ generic $n\times n$ matrices. By adjoining to it the traces of all its elements, we obtain the trace ring $\Pi_{m,n}$ of $m$ generic $n\times n$ matrices; see for example [1], [5]. If $\deg(x_{ij}(l))=1$ for all $i,j$ and $l$, then $\Pi_{m,n}$ is a positively graded $k$-algebra $\bigoplus_{i=0}^{\infty}(\Pi_{m,n})_i$. Its Poincaré series is then the formal power series over $\mathbf{Z}$:

$$\mathcal{P}(\Pi_{m,n};t)=\sum_{i=0}^{\infty}\dim_k((\Pi_{m,n})_i).t^i.$$

Similarly, if $\deg(X_{ij}(1))=(0,\ldots,1,\ldots,0)=e_1$, then $\Pi_{m,n}$ is a $\mathbf{N}^{(m)}$-graded $k$-algebra. Its Poincaré series in this multigradation is then:

$$\mathcal{P}(\Pi_{m,n};t_1,\ldots,t_m)=\sum_{(i_1,\ldots,i_m)}\dim_k((\Pi_{m,n})_{(i_1,\ldots,i_m)}).t_1^{i_1}\cdots t_m^{i_m}.$$

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For $n = 2$, a power series expansion for the Poincaré series was given by C. Procesi [7] and in the multigradation by E. Formanek [2].

We will now give a rational expression for $\mathcal{P}(\Pi_{m,2}; t_1, \ldots, t_m)$ using some 60-year-old results due to H. Weyl [10, p. 11 and p. 17] and I. Schur [8].

Let $R_{m,n}$ denote the center of $\Pi_{m,n}$. In [6] Procesi has proved that the map

$$f \mapsto \mathrm{Tr}(f . X_{m+1})$$

defines a monomorphism from $\Pi_{m,n}$ onto the subspace of $R_{m+1,n}$ consisting of all elements of degree one in $X_{m+1}$, i.e. $\Sigma_{(i_1,\ldots,i_m)}(R_{m+1,n})_{(i_1,\ldots,i_m,1)}$. Translating to Poincaré series, this means that $\mathcal{P}(\Pi_{m,n}; t_1, \ldots, t_m)$ is the coefficient of $t_{m+1}$ in the power series expansion of $\mathcal{P}(R_{m+1,n}; t_1, \ldots, t_m, t_{m+1})$, or equivalently:

$$\mathcal{P}(\Pi_{m,n}; t_1, \ldots, t_m) = \frac{\partial}{\partial t_{m+1}} \mathcal{P}(R_{m+1,n}; t_1, \ldots, t_m, t_{m+1})\big|_{t_{m+1}=0}.$$

If $n = 2$ one can give a fairly precise description of $R_{m+1,2}$, [7]. Recall from [7] that $R_{m+1,2}$ is the ring of invariant polynomial mapping from $m + 1$ copies of $M_2(k)$ under the componentwise action by conjugation of $\mathrm{GL}_2(k)$. Now, $M_2(k)$ decomposes in the direct sum $k \oplus M^0$ with $M^0$ the 3-dimensional vector space of trace zero matrices. Therefore, $R_{m+1,2}$ is the polynomial ring in the elements $\mathrm{Tr}(X_1), \ldots, \mathrm{Tr}(X_{m+1})$ over the ring of invariants of $m + 1$ copies of $M^0$ under induced action of $6L_2(k)$, $R^0_{m+1,2}$.

$M$ is endowed with the nondegenerate quadratic form $\mathrm{Tr}(A^2)$, thus $\mathrm{GL}_2(k)$ acts on $M^0$ inducing the full group $\mathrm{SO}(M^0)$ of special orthogonal transformations for the form $\mathrm{Tr}(A^2)$. Therefore, $R^0_{m+1,2}$ is the ring of special orthogonal invariants of $m + 1$ copies of the standard representation.

The composition of the Poincaré series in the multigradation of this ring was carried out by H. Weyl [10, p. 17] and I. Schur [8]. They found the following rational expression:

$$\mathcal{P}(R^0_{m+1,2}; t_1, \ldots, t_{m+1}) = \frac{[1 + t^{2m-1} \cdot \cdots \cdot t^{m-2} + t^{m+1}; t^{m-1}, t^m]}{\Pi_{i<k}^{m+1} (t_k - t_i) . \Pi_{i \leqq k}^{m+1} (1 - t_i t_k)}$$

where the numerator of this expression denotes the determinant of the following $(m + 1) \times (m + 1)$ matrix:

$$\begin{bmatrix} 1 + t_1^{2m-1} & 1 + t_2^{2m-1} & \cdots & 1 + t_{m+1}^{2m-1} \\ t_1 + t_1^{2m-2} & t_2 + t_2^{2m-2} & \cdots & t_{m+1} + t_{m+1}^{2m-2} \\ \vdots & \vdots & & \vdots \\ t_1^{m-2} + t_1^{m+1} & t_2^{m-2} + t_2^{m+1} & \cdots & t_{m+1}^{m-2} + t_{m+1}^{m+1} \\ t_1^{m-1} & t_2^{m-1} & \cdots & t_{m+1}^{m-1} \\ t_1^m & t_2^m & \cdots & t_{m+1}^m \end{bmatrix}$$

Combining the facts we get:

$$(*)\quad \mathcal{P}(\Pi_{m,2};t_1,\ldots,t_m)=\frac{\partial}{\partial t_{m+1}}\frac{[1+t^{2m-1},\ldots,t^{m-2}+t^{m+1};t^{m-1},t^m]}{\Pi_{j=1}^{m+1}(1-t_j)\Pi_{i<k}^{m+1}(t_k-t_i)\Pi_{i\leq k}^{m+1}(1-t_it_k)}\bigg|_{t_{m+1}=0}.$$

Calculating the numerator of the right hand side of (*) gives:

$$\prod_{j=1}^{m}(1-t_j).\prod_{i<k}^{m}(t_k-t_i)\cdot(-1)^m.\prod_{j=1}^{m}t_j.\prod_{i\leq k}^{m}(1-t_it_k).M_1$$

$$-\Big\{-\prod_{j=1}^{m}(1-t_j).\prod_{i<k}^{m}(t_k-t_i).(-1)^m.\prod_{j=1}^{m}t_j.\prod_{i\leq k}^{m}(1-t_it_k)$$

$$+\prod_{j=1}^{m}(1-t_j).\prod_{i<k}^{m}(t_k-t_i).\Big(\sum_{j=1}^{m}(-1)^{m-1}t_1\cdots\check{t}_j\cdots t_m\Big).\prod_{i\leq k}^{m}(1-t_it_k)$$

$$+\prod_{j=1}^{m}(1-t_j).\prod_{i<k}^{m}(t_k-t_i).(-1)^m.\prod_{j=1}^{m}t_j.\prod_{i\leq k}^{m}(1-t_it_k).\Big(\prod_{j=1}^{m}-t_j\Big)\Big\}.M_2$$

where $M_1$ and $M_2$ are defined to be:

$$M_1=\det\begin{bmatrix}1+t_1^{2m-1} & \cdots & 1+t_m^{2m-1} & 0\\ & & & 1\\ \vdots & & \vdots & 0\\ & & & \vdots\\ t_1^{m-1}+t_1^{m+1} & \cdots & t_m^{m-1}+t_m^{m+1} & 0\\ t_1^{m-1} & \cdots & t_m^{m-1} & 0\\ t_1^m & \cdots & t_m^m & 0\end{bmatrix}$$

$$=(-1)^{m+2}.\det\begin{bmatrix}1+t^{2m-1} & \cdots & 1+t_m^{2m-1}\\ t_1^2+t_1^{2m-3} & \cdots & t_m^2+t_m^{2m-3}\\ \vdots & & \vdots\\ t_1^{m-1}+t^{m+1} & \cdots & t_m^{m-1}+t_m^{m+1}\\ t_1^{m-1} & \cdots & t_m^{m-1}\\ t_1^m & \cdots & t_m^m\end{bmatrix}=(-1)^{m+2}.\Delta_1,$$

$$M_2=\det\begin{bmatrix}1+t_1^{2m-1} & \cdots & 1+t_m^{2m-1} & 1\\ \vdots & & \vdots & 0\\ & & & \vdots\\ t_1^{m-1}+t_1^{m+1} & \cdots & t_m^{m-1}+t_m^{m} & 0\\ t_1^{m-1} & \cdots & t_m^{m-1} & 0\\ t_1^m & \cdots & t_m^m & 0\end{bmatrix}$$

$$= (-1)^{m+1} . \det \begin{bmatrix} t_1 + t_1 & \cdots & t_m + t_m^{2m-1} \\ \vdots & & \vdots \\ t_1^{m-1} + t_1^{m+1} & \cdots & t_m^{m-1} + t_m^{m+1} \\ t_1^{m-1} & \cdots & t_m^{m-1} \\ t_1^m & \cdots & t_m^m \end{bmatrix} = (-1)^{m+1} . \Delta_2 .$$

Therefore, the numerator is equal to:

$$\prod_{j=1}^{m} (1 - t_j) . \prod_{i<k}^{m} (t_k - t_i) . \prod_{i \leq k}^{m} (1 - t_i t_k) [e_m . \Delta_1 - (e_m + e_{m-1} + e_1 e_m) . \Delta_2]$$

where $e_i$ denotes the $i$th elementary symmetric function in $m$ variables. This finishes the proof of:

THEOREM 1. *The Poincaré series of the trace ring of m generic* $2 \times 2$ *matrices has the following rational expression*:

$$\mathcal{P}(\Pi_{m,2}, t_1, \ldots, t_m) = \frac{e_m . \Delta_1 - (e_m + e_1 . e_m + e_{m-1}) . \Delta_2}{e_m^2 . \Pi_{j=1}^m (1 - t_j) . \Pi_{i<k}^m (t_k - t_i) . \Pi_{i \leq k}^m (1 - t_i t_k)} \quad (m > 2).$$

## 2. The functional equation

The main result of this section is:

THEOREM 2. *The Poincaré series of the trace ring of m generic* $2 \times 2$ *matrices satisfies the functional equation*:

$$\mathcal{P}(\Pi_{m,2}, 1/t) = -t^{4m} . \mathcal{P}(\Pi_{m,2}; t) \qquad (m > 2).$$

PROOF. We note that:

$$e_m^{2m-1} . \Delta_1 \left( \frac{1}{t_1}, \ldots, \frac{1}{t_m} \right) = -\Delta_1(t_1, \ldots, t_m),$$

$$e_m^{2m-1} . \Delta_2 \left( \frac{1}{t_1}, \ldots, \frac{1}{t_m} \right) = -\Delta_2(t_1, \ldots, t_m),$$

$$e_m^2 . \left( e_m + e_1 e_m + e_{m-1} \left( \frac{1}{t_1}, \ldots, \frac{1}{t_m} \right) \right) = e_m + e_1 . e_m + e_{m-1},$$

and therefore we get:

$$\mathcal{P}\left( \Pi_{m,2}; \frac{1}{t_1}, \ldots, \frac{1}{t_m} \right) = \frac{-e_m^{-2m-1} . \{ e_m \Delta_1 - (e_m + e_1 e_m + e_{m-1}) . \Delta_2 \}}{e_m^{-2m-5} . \{ \Pi_{j=1}^m (t_j - 1) . \Pi_{i<h}^m (t_i - t_k) . \Pi_{i \leq h}^m (t_i t_k - 1) \}}$$

$$= -e_m^4 . (-1)^{2m + m^2 - m} . \mathcal{P}(\Pi_{m,2}; t_1, \ldots, t_m).$$

Finally, specializing $t_1 = t_2 = \cdots = t_m = t$, we get the desired result.

In [3] we have shown that there exists an iterated Ore-extension $\Gamma_m$ and a natural morphism

$$\pi_m : \Gamma_m \twoheadrightarrow \Pi_{m,2}.$$

In analogy with the commutative case, we say that $\Pi_{m,2}$ is a Gorenstein ring iff

$$\operatorname{Ext}^i_{\Gamma_m}(\Pi_{m,2}, \Gamma_m) = 0 \qquad \text{for } i \neq 0, \frac{(m-2)(m-3)}{2},$$

$$\operatorname{Ext}^j_{\Gamma_m}(\Pi_{m,2}, \Gamma_m) \cong \Pi_{m,2} \qquad \text{for } j = \frac{(m-2)(m-3)}{2}.$$

In [4] it is shown that Gorensteinness of $\Pi_{m,2}$ is equivalent to the following:

CONJECTURE. *$\Pi_{m,2}$ is a Cohen–Macauley module.*

This fact is very plausible, since $\Pi_{m,2}$ is the fixed module of a free module over a regular domain under a reductive group.

As in the commutative case [9], it would follow from the fact that $\Pi_{m,2}$ is Gorenstein that its Poincaré series satisfies the functional equation

$$\mathcal{P}\left(\Pi_{m,2}; \frac{1}{t}\right) = (-1)^{k\dim(\Pi_{m,2})} . t^{\alpha} . \mathcal{P}(\Pi_{m,2}; t)$$

for some $\alpha \in \mathbf{Z}$.

Because $K\dim(\Pi_{m,2}) = 4m - 3$, Theorem 2 supports the above conjecture.

*Added in proof* (December 1984). The above conjecture has been verified by the author. Details will appear elsewhere, [4].

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